The Russian Improvement

The Russian probablist Andrei Markov made some improvements to Bernoulli's theorem . One of the improvements may be stated as:

If $\frac{W_{NR}}{W_{NR+N}} > C + 1$ then $\frac{W_{NR+1} + W_{NR+2} + \ldots + W_{NR+N}}{W_{NR+N+1} + W_{NR+N+2} + \ldots + W_{NT}} > C$

and if $\frac{W_{NR}}{W_{NR-N}} > C + 1$ then $\frac{W_{NR-1} + W_{NR-2} + \ldots + W_{NR-N}}{W_{NR-N-1} + W_{NR-N-2} + \ldots + W_0} > C$

In this paper I give a derivation of this improvement.

In my paper <u>Bernoulli's Theorem</u> in lemma 4 , the chain of inequalities of the lemma stops at $\frac{W_{NR+N}}{W_{NR+2N}}$ but it could be continued to $\frac{W_{NT-N}}{W_{NT}}$ and since $\frac{W_{NR}}{W_{NR+N}}$ is less than any of the ratios in the chain, we

get using lemma 5 the following inequalities:

$$\frac{W_{NR}}{W_{NR+N}} < \frac{W_{NR+1} + W_{NR+2} + \ldots + W_{NR+N}}{W_{NR+N+1} + W_{NR+N+2} + \ldots + W_{NR+2N}}$$

$$\frac{W_{NR}}{W_{NR+N}} < \frac{W_{NR+N+1} + \ldots + W_{NR+2N}}{W_{NR+2N+1} + \ldots + W_{NR+3N}}$$

$$\frac{W_{NR}}{W_{NR+N}} < \frac{W_{NR+2N+1} + \ldots + W_{NR+3N}}{W_{NR+3N+1} + \ldots + W_{NR+4N}}$$

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$$\frac{W_{NR}}{W_{NR+N}} < \frac{W_{NR+(S-2)N+1} + \ldots + W_{NT-N}}{W_{NT-N+1} + \ldots + W_{NT}}$$

Taking the reciprocals of both sides of the inequalities, the inequality sign reverses and we get:

1) $\frac{W_{NR+N}}{W_{NR}} > \frac{W_{NR+N+1} + W_{NR+N+2} + \ldots + W_{NR+2N}}{W_{NR+1} + W_{NR+2} + \ldots + W_{NR+N}}$

2) $\frac{W_{NR+N}}{W_{NR}} > \frac{W_{NR+2N+1} + W_{NR+2N+2} + \ldots + W_{NR+3N}}{W_{NR+N+1} + W_{NR+N+2} + \ldots + W_{NR+2N}}$

3) $\frac{W_{NR+N}}{W_{NR}} > \frac{W_{NR+3N+1} + W_{NR+3N+2} + \ldots + W_{NR+4N}}{W_{NR+2N+1} + W_{NR+2N+2} + \ldots + W_{NR+3N}}$

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The denominator of each fraction on the right side of the above list of inequalities is equal to the numerator of the fraction just above it, so if we multiply the first K rows of inequalities, the numerators and denominators on the right side will all cancel each other except for the denominator in the first row and the numerator in the Kth row.

So $\left(\frac{W_{NR+N}}{W_{NR}}\right)^{K} > \frac{W_{NR+KN+1} + W_{NR+KN+2} + \ldots + W_{NR+(K+1)N}}{W_{NR+1} + W_{NR+2} + \ldots + W_{NR+N}}$

So $\left(\frac{W_{NR+N}}{W_{NR}}\right) + \left(\frac{W_{NR+N}}{W_{NR}}\right)^{2} + \left(\frac{W_{NR+N}}{W_{NR}}\right)^{3} + \ldots + \left(\frac{W_{NR+N}}{W_{NR}}\right)^{(S-1)} > \frac{W_{NR+N+1} + \ldots + W_{NT}}{W_{NR+1} + \ldots + W_{NR+N}}$

So since $\left(\frac{W_{NR+N}}{W_{NR}}\right) / \left[1-\frac{W_{NR+N}}{W_{NR}}\right] = \left(\frac{W_{NR+N}}{W_{NR}}\right) + \left(\frac{W_{NR+N}}{W_{NR}}\right)^2 + \left(\frac{W_{NR+N}}{W_{NR}}\right)^3 + \ldots$

We get $\left(\frac{W_{NR+N}}{W_{NR}}\right) / \left[1-\frac{W_{NR+N}}{W_{NR}}\right] > \frac{W_{NR+N+1}+....+W_{NT}}{W_{NR+1}+....+W_{NR+N}}$.

If $\left(\frac{W_{NR+N}}{W_{NR}}\right) / \left[1-\frac{W_{NR+N}}{W_{NR}}\right] = 1/C$ then

$1/C > \frac{W_{NR+N+1}+....+W_{NT}}{W_{NR+1}+....+W_{NR+N}}$, so $\frac{W_{NR+1}+....+W_{NR+N}}{W_{NR+N+1}+....+W_{NT}} > C$.

Solving the equation $\left(\frac{W_{NR+N}}{W_{NR}}\right) / \left[1-\frac{W_{NR+N}}{W_{NR}}\right] = 1/C$ for $\frac{W_{NR}}{W_{NR+N}}$ we get

$\frac{W_{NR}}{W_{NR+N}} = C + 1$. So if $\frac{W_{NR}}{W_{NR+N}} = C + 1$, $\frac{W_{NR+1}+....+W_{NR+N}}{W_{NR+N+1}+....+W_{NT}} > C$.

Using the same method we get

If $\frac{W_{NR}}{W_{NR-N}} = C + 1$ then $\frac{W_{NR-1}+....+W_{NR-N}}{W_{NR-N-1}+....+W_{0}} > C$

Using this improvement with Bernoulli's method, the NT calculated for Bernoulli's example drops from 25,550 to 17,350.
Using my method, it drops from 15,200 to 10,550.